华东师范大学期末试卷

2012－2013 学年第一学期

课程名称: Advanced ODE (International Students)

学生姓名:____________________ 学　　号:________________

专　　业:__Applied Math__ 年级/班级:__Grade One_____

课程性质：Required Course (Core) (Open Exam from 2012/12/25 to 2013/01/07)

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Question 1. (20 Marks) Consider the following IVP:

$$x' = f(t,x),\ \ x(0)=0,$$

where

$$f(t,x)=\begin{cases} 0, & t\le 0,\ -\infty<x<\infty \\ 2t, & t>0,\ x<0 \\ 2t-\dfrac{4}{t}x, & t>0,\ 0\le x<t^2 \\ -2t, & t>0,\ t^2\le x<\infty \end{cases}.$$

1. To show $\lim\limits_{\substack{t\to 0\\ x\to 0}} f(t,x)=0=f(0,0)$;

2. To show that $f(t,x)$ is continuous on R^2;

3. To show that $f(t,x)$ doesn't satisfy a Lipschitz condition in any neighborhood of

the origin by showing $\lim\limits_{\substack{t\to 0\\ x\to 0}} \dfrac{\partial f}{\partial x}(t,x)=\infty$ (why? Give the detail).

4. To construct by Picard iteration with $x_0(t)\equiv 0$ to get the Picard sequence given by

$$x_{2m-1}(t)=t^2,\ \ x_{2m}(t)=-t^2,\ \ m\in N^+,$$

Is this Picard sequence $\{x_n(t)\}$ convergent uniformly on $t\in[-h,h]$ or not? Give your reasoning.

5. Although its two subsequences do converge uniformly on $t\in[-h,h]$, are they convergent uniformly to the solution? Give your reasoning.

6. However, the uniqueness of solution is still there. Give your judgment on what implication can be concluded by this phenomenon.
(The proof of uniqueness is given as follows.

If there exist two solutions $x_1(t)$ and $x_2(t)$, defined on $t \in [0, h]$, where $0 < h < \infty$, then $\delta(t) = (x_1(t) - x_2(t))^2$ satisfies

$$\delta(0) = 0, \quad \delta(t) \geq 0, \quad t \in [0, h].$$

Then

$$\delta'(t) = 2[x_1'(t) - x_2'(t)][x_1(t) - x_2(t)] = 2[f(t, x_1(t)) - f(t, x_2(t))][x_1(t) - x_2(t)].$$

Since $f(t,x)$ is not increased on x by the definition of $f(t,x)$, we have

$$[f(t, x_1(t)) - f(t, x_2(t))][x_1(t) - x_2(t)] \leq 0, \quad t \in [0, h].$$

It yields $\delta'(t) \leq 0$, which implies $\delta(t) \leq 0$, $t \in [0, h]$. Then it must have $\delta(t) \equiv 0$ for $t \in [0, h]$. That is, $x_1(t) \equiv x_2(t)$ for $t \in [0, h]$. It is similar to show for $t \in [-h, 0]$. The uniqueness is done.)

7. Can you find this unique solution of the IVP? Try it.

Question 2. (20 Marks) Show that the continuous map $H : R^3 \to R^3$ defined by

$$H(x) = \begin{pmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + \dfrac{x_1^2}{3} \end{pmatrix}$$

has a continuous inverse $H^{-1} : R^3 \to R^3$ and that the nonlinear system $x' = f(x)$ with

$$f(x) = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}$$

is transformed into the linear system $x' = Ax$ with $A = Df(0)$ under the map; i.e.

if $y = H(x)$, show that $y' = Ay$.

Question 3. (20 Marks)

1. Find the first three successive approximations $u^{(1)}(t;a)$, $u^{(2)}(t;a)$ and $u^{(3)}(t;a)$ for

$$\begin{cases} x_1' = -x_1 \\ x_2' = x_2 + x_1^2 \end{cases}$$

and use $u^{(3)}(t;a)$ to approximate the stable manifold S near the origin. Also approximate the unstable manifold near the origin for this system. Note that $u^{(3)}(t;a) = u^{(2)}(t;a)$ and therefore $u^{(j+1)}(t;a) = u^{(j)}(t;a)$ for $j \geq 2$. Thus

$$u^{(j)}(t;a) \to u(t;a) = u^{(2)}(t;a),$$

which gives the exact function defining S.

2. Solve the above system and show that S and U are given by

$$S: x_2 = -\frac{x_1^2}{3};\ \ U: x_1 = 0.$$

Sketch S, U, E^S and E^U.

Question 4. (20 Marks)

1. State LaSalle-Krosovskii's Theorem.
2. Apply LaSalle-Krosovskii's Theorem to the following general pendulum equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - h(x_2) \end{cases},$$

where $g(\cdot)$ and $h(\cdot)$ are locally Lip. and satisfy

$$g(0) = 0,\ \ yg(y) > 0,\ \ \forall\ y \neq 0,\ \ y \in (-\infty, \infty);$$

$$h(0) = 0,\ \ yh(y) > 0,\ \ \forall\ y \neq 0,\ \ y \in (-\infty, \infty),$$

to conclude its stability of the origin.

Questions 5. (20 Marks) The system of the form given by

$$x' = -\nabla V(x)$$

is called a gradient system, where $V: D \to R$ is of C^2, $D \subset R^n$.

Let $V: R^2 \to R$ be a function with

$$V(x_1, x_2) = 2x_1^2(x_1 - 1) + \frac{x_2^2}{2}.$$

1. Show that

$$\begin{cases} x_1' = -2x_1(x_1 - 1)(2x_1 - 1) \\ x_2' = -x_2 \end{cases}$$

   is a gradient system.

2. Find all possible equilibriums of the above gradient system. Determining the stability of each equilibrium.